Proposed central limit behavior in deterministic dynamical systems

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We check claims for a generalized central limit theorem holding at the Feigenbaum (infinite bifurcation) point of the logistic map made recently by Tirnakli *et al.*, Phys. Rev. **75**, 040106(R) (2007); this issue, Phys. Rev. **79**, 056209 (2009). We show that there is no obvious way that these claims can be made consistent with high statistics simulations. Instead, we find other scaling laws for related quantities.

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One of the main results in mathematical statistics is the Gaussian central limit (CL) theorem. It holds for sequences of i.i.d. (independent identically distributed) random variables with finite variance and says that a sum over N such variables tends for $N\to\infty$, after shifting by an amount $O(N)$ and rescaling by $N^{-1/2}$, to a Gaussian. Similar CL theorems hold for i.i.d. variables with infinite variance, for which the limit distribution is under suitable conditions Levy stable [1].

Actually, the conditions for which these CL theorems hold can be substantially relaxed. They still hold for correlated variables provided the correlations are short range, and at least the Gaussian CL theorem holds also stationary chaotic processes, i.e., when successive variables are related by a deterministic equation $x_{i+1}=F(x_i)$, provided the map $F(x)$ is mixing and has positive entropy [2].

In two recent papers [3,4], it was claimed that this well-known result can be generalized to maps with zero entropy which lead to sequences with long-range correlations. In particular, these authors studied in [3] the logistic map at the Feigenbaum (infinite bifurcation [5]) point. In [4] they studied this map *near* (but not *at*) the Feigenbaum point. Although their results in [4] contradict those in [3] and although they actually did not study CL behavior at all in Ref. [4], they did not revoke their previous claim and concluded again that an anomalous CL theorem holds at the Feigenbaum point. In the following, I will try to clarify this problem.

Let us consider trajectories $x_{i+1}=f_a(x_i)$ of length N, with $f_a(x)=a-x^2$ and with randomly distributed x_0. The Feigenbaum point is at $a=a_c\equiv 1.401\,155\,189\,092\,050\,6\ldots$. Following [3], we study sums $Y=\Sigma_{i=N_0+1}^{N_0+N}x_i$ and their distributions at $a=a_c$ and for large N. Here, N_0 is the length of a possible discarded transient.

Denoting by $\langle Y\rangle$ the average over x_0, the claim in [3] is that the centered and suitably rescaled sums

$$y=N^{\gamma}(Y-\langle Y\rangle), \tag{1}$$

are distributed according to a "q Gaussian"

$$p(y)\propto\frac{1}{[c+y^2]^b} \tag{2}$$

for $\gamma=1.5$, with $b\approx 4/3$ and $c\approx 0.1$ [6]. Moreover, it is claimed that the same distribution, with identical γ, b, and c, is found for the modified logistic map $f_{a,z}(x)=a-x^z$ with $z=1.75$ and $z=3$. If true, this universality would be remarkable. In a subsequent publication [4], the authors presented additional details which supposedly supported these claims. Unfortunately, none of them seem to be correct.

Transients are not discussed in [3] (but in [4]). Since the transient dynamics of the Feigenbaum map at $a=a_c$ is universal [5,7] and since using $N_0>0$ might ruin the scale invariance by introducing a new time scale, one might conclude that $N_0=0$ was used there. But this is not true (private communication) and transients have actually been discarded both in [3,4].

Let us nevertheless start by discussing first the case $N_0=0$. It is straightforward to do the necessary simulations to estimate $p(y)$. In all simulations, x_0 was uniformly distributed in $[0,a]$. Results, for several values of a at and slightly above a_c, are shown in Fig. 1. Indeed, in this figure, histograms of the nonrescaled and nonshifted sums Y, for $N=16384$, are shown. Similar results were obtained for other values of N. They have markedly different behavior left and right of the central peak $Y_c\approx 86\,333$. For $Y<Y_c$, we observe a very steep rise $P(Y)\sim e^{0.86Y}$, while the decrease for $Y>Y_c$ is much more gentle $P(Y)\sim e^{-0.17Y}$. Superposed on both exponentials are periodicities which obviously result from the hierarchical structure of the Feigenbaum attractor.

Figure 1 is very different from the results shown in [3], suggesting that indeed transients have been discarded there. We stress again that this would not have been needed for obtaining universal results and that the results shown in Fig. 1 are universal.

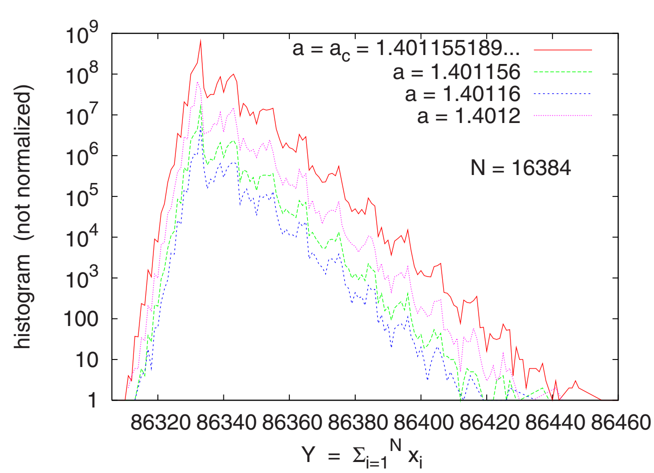

FIG. 1. (Color online) Histograms of Y for $N=16\,384$ and for four values of a at or slightly above criticality.

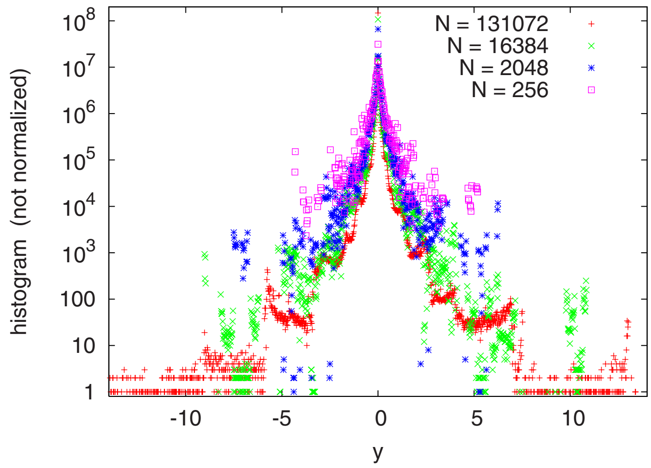

FIG. 2. (Color online) Histograms of y for $a=a_c$, $N_0=52\,4288$, and for $N=256$, 2048, 16384, and 131072. Notice that for all curves $N_0 \gg N$, that for all curves the accuracy of a_c should be sufficient, and that round-off errors should be negligible.

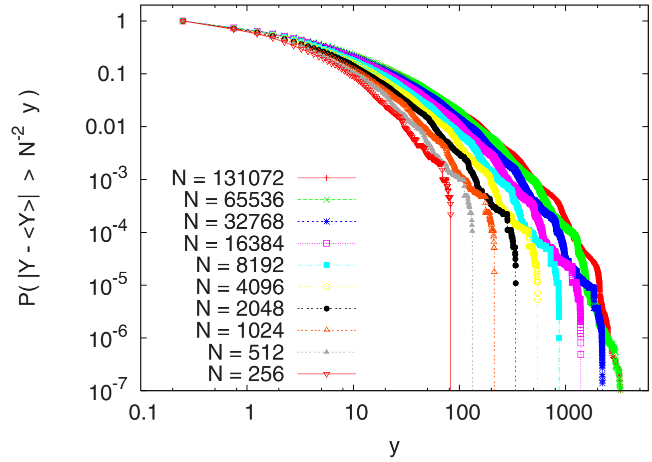

FIG. 3. (Color online) Cumulative distributions corresponding to Fig. 2 (and to similar curves for other values of N) plotted against $\ln y$, with $\gamma=2$. Without changing the value of the exponent γ from 1.5 to 2, the curves would collapse even less.

In the rest of this Brief Report, we shall discuss the case $N_0>0$. If one does so and hopes for scale invariance, one has to take $N_0 \gg N$. Otherwise one introduces a time scale in the supposed scaling regime. At least for one simulation it is mentioned in [4] that $N_0=4096$ was used. In lack of better information, we shall assume that the same N_0 was used also in all other simulations of [3,4], although it does not satisfy the criterium $N_0 \gg N$. Moreover, the 12-digit approximation $a_c \approx 1.401\,155\,189\,092$ is cited in [4], and we assume that the same approximation was used in [3]. With these parameters, we were able to reproduce Fig. 4 of [3], but it is clear that one should be careful in accepting this as the true limit behavior. We have therefore repeated these simulations with much longer transients ($N_0=2^{19}$), higher precision for a_c (18 digits [8]), and higher statistics ($>10^8$ starting values).

The results shown in Fig. 2 are qualitatively similar to Fig. 4 of [3], but they are definitely not described by Eq. (2), even if we disregard the strong fluctuations resulting from the structure of the Feigenbaum attractor (see also Fig. 5). In order to minimize the visual effect of these fluctuations, we plot in Fig. 3 the cumulative distributions. We also replace the value 1.5 of the exponent γ [see Eq. (1)] by $\gamma=2$. We see that these cumulative distributions are reasonably well fitted by

$$P(y) = \int_y^\infty dx p(x) \approx e^{-\phi(\ln y/N^\beta)}, \tag{3}$$

with a quadratic scaling function $\phi(z)$ and with $\beta \approx 0.05$.

Although the results of Ref. [4] supposedly confirm the claim of q Gaussianity made in [3], a very different limit is actually considered in Ref. [4]. Instead of using $a=a_c$ and $N_0 \gg N \to \infty$, the authors there consider the case where N_0 is not much larger than N, where $a>a_c$, and

$$N \sim (a-a_c)^{-2\Delta} \gg 1. \tag{4}$$

Here $\Delta = \ln 2/\ln \delta$, and $\delta=4.669\ldots$ is one of the Feigenbaum constants [5]. In this region the Feigenbaum attractor consists of $n=2^k$ "bands" with $k \approx (a-a_c)^{1/\delta}$ [5,7]. Orbits on it jump periodically between the bands but are chaotic within each band. On each band, the n-fold iterated map $f_a^{(n)}$ is mixing. Thus Y is a sum over n series of random variables, each of which shows normal CL behavior for $N \to \infty$. Therefore, Y also shows normal CL behavior in the limit $N\to\infty$, n=const. It is for this reason that this limit is replaced in [4] by $N \propto n^2 \to \infty$. Strictly speaking, we are then no longer dealing with (normal or abnormal) the central limit behavior at all. Nevertheless, it is of interest to study the asymptotic behavior.

Before we go to the numerics, we should point out that the limit $N \propto n^2 \to \infty$ corresponds, in renormalization-group language, to a cross-over region between two different scaling limits where $a>a_c$ (Gaussian) and $a=a_c$ (non-Gaussian illustrated by Figs. 2 and 3). In general, one does not have simple analytic behavior in such cross-over regions. If it is true, as is claimed in [3,4], that the behavior is given—at least for one particular value of N/n^2—by a simple formula such as Eq. (2), this would be extremely surprising.

In the following we shall, for definiteness, only deal with band-merging points, where $n=2^k$ bands merge into $n/2$ bands as a is increased; but similar behavior is found also for other values of a. Indeed, when looking at distributions of y for large n, $N \gg n$, and $N_0 \gg n$, one finds heavy-tailed distributions (see Fig. 4). But as closer inspection shows, they are in general not described by Eq. (2) (see Fig. 5). Apart from the steps and discontinuities at large y which might recede to infinity in the limit indicated above, the main deviations are (i) a systematic downward curvature in Fig. 5 for intermediate to large y, seen most clearly in the curve for $n=4096$; (ii) deviations from straight-line behavior at very small y, both for $n \ll \sqrt{N}$ and for $n \gg \sqrt{N}$. In a narrow region of N/n^2, the data are in *rough* agreement with Eq. (2) for small y, but there is no value of N/n^2 where a linear fit in Fig. 5 would be acceptable.

In view of this, it seems very unlikely that the rough agreement with Eq. (2) is more than a numerical coincidence. The data shown in Figs. 4 and 5 of [3] are definitely

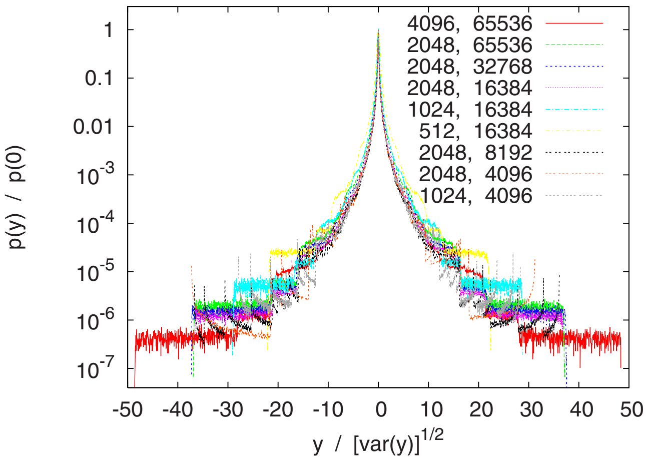

FIG. 4. (Color online) Distributions of $y/\sqrt{\mathrm{var}(y)}$, normalized to $p(0)=1$, for various values of N and n, where a is set to the $n \to n-1$ band-merging point. In all cases, $N_0 \geq 16\,384$. Similar results were obtained also for other values of a. Notice that the statistics in any curve of this figure is at least ten times higher than in any of the curves in [3,4].

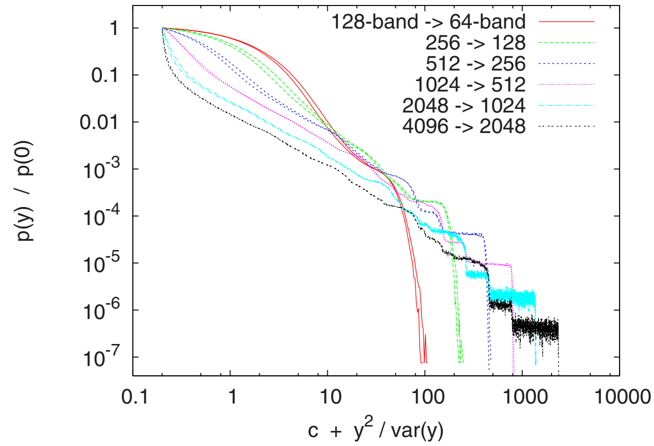

FIG. 5. (Color online) Distributions of $y/\sqrt{\mathrm{var}(y)}$ for $N=65\,536$ normalized to $p(0)=1$ plotted against y^2+const on a log-log plot. As in Fig. 4, long transients ($N_0=65\,536$ in most cases) have been discarded, and the control parameter a of the logistic map is chosen as the $n \to n/2$ band-merging point. According to Eq. (2), one would expect straight lines with slopes $-b$. Apart from the rather unsystematic deviations at large y which could be effects which vanish in the limit $N \gg n \to \infty$, one sees a systematic downward curvature for intermediate y and strong systematic upward (downward) curvatures for $n > \sqrt{N}(n < \sqrt{N})$ at very small y. Notice that one has two curves for each n: one for $y>0$ and one for $y<0$. The case $a=a_c$ plotted differently in Fig. 3 corresponds to the limit $n \to \infty$.

not well fitted by Eq. (2) (not even for very small y/σ). We might add that equally good (or bad) fits would be obtained with Levy stable distributions [1], which moreover might have more theoretical justification. It might be that the sums leading to Y can be reformulated by splitting them into two steps, $Y=\Sigma_i x_i = \Sigma_{\mathcal{I}} \Sigma_{i \in \mathcal{I}} x_i$, such that the partial sums $\eta_{\mathcal{I}} = \Sigma_{i \in \mathcal{I}} x_i$ are weakly correlated and have heavy-tailed distributions. In view of the above remarks about cross overs, we do not consider this as very likely, but even such a remote possibility of justification seems absent for Eq. (2).

Some final remarks: (i) the behavior described here is seen only when N is a power of 2. Otherwise, one observes completely different behavior. (ii) The fluctuations of Y are, for $N_0, N \gg n \gg 1$, tiny. All structures shown in Figs. 4 and 5 (including the tails) extend, before centering and multiplying by N^{γ}, over a range $\Delta Y < 10^{-3}$. For $N=65\,536$, this is to be compared to $\langle Y \rangle \approx 34\,533$, i.e., all relative fluctuations are smaller than 3×10^{-8} [9]. The reason for this is that the motion on an n-band attractor with large n is extremely regular, with the chaos confined to very narrow bands. Thus, if a generalized CL theorem holds for this problem in any sense, it is completely unobservable in any experimental situation.

(iii) Since the phenomenon illustrated in Figs. 4 and 5 seems to describe corrections to the scaling limit of the Feigenbaum map, it is not clear how much it depends on the original map where one starts from and on the distribution of x_0. The only phenomenon discussed in this Brief Report which has a realistic chance to be experimentally accessible and is likely to be universal is the behavior shown in Fig. 1. It is dominated by chaotic transients and is very far from anything described in Refs. [3,4].

[1] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Macmillan, London, 1982).

[2] T. Geisel and J. Nierwetberg, Phys. Rev. Lett. **48**, 7 (1982).

[3] U. Tirnakli, C. Beck, and C. Tsallis, Phys. Rev. E **75**, 040106(R) (2007).

[4] U. Tirnakli, C. Tsallis, and C. Beck, this issue, Phys. Rev. E **79**, 056209 (2009).

[5] M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978); J. Stat. Phys. **21**, 669 (1979).

[6] In the q-statistics community, Eq. (2) is usually written in a more cumbersome way. We prefer the simpler and more intuitive form given here.

[7] P. Grassberger and M. Scheunert, J. Stat. Phys. **26**, 697 (1981).

[8] All simulations in the present Brief Report used the GCC compiler on an INTEL platform, with floating point operations performed with "long double" (80 bit) data type. Tests were made with standard double precision (64 bit).

[9] This again raises the question of numerical precision [8].